# A Journey Guided by the Stars - Recap

Core Model Semiar, CMU

Andreas Lietz

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## Contents



In Part I, we give some historical context of saturation properties of ideals on $\omega_1$ and the nonstationary ideal in particular. Specifically, we look at precipitous, saturated and $\omega_1$-dense ideals. We also give a (very!) brief introduction to $\mathbb{P}_{\max}$ and $\mathbb{Q}_{\max}$.

In Part II, we motivate the strategy we will follow to force "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense" from large cardinals. We have to develop a technique which allows iterating forcings which are not stationary set preserving without collapsing $\omega_1$. One key idea here is that one should not kill "old" stationary sets. We introduce a new class of forcings, the respectful forcings which roughly play the role of semiproperness in the main iteration theorem.

In Part III, we deal with the key $\Pi_1$-property we have to preserve along the iteration, namely a witness to $\diamondsuit(\omega_1^{<\omega})$. We look more closely at associated classes of forcings, loosely called $\diamondsuit$-forcings, and introduce the forcing axiom QM which implies "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense". We finally consider version of the Asperó-Schindler-forcing that we can understand as "the sealing forcing for $\omega_1$-density".

# 0 Part I

We prove the following theorem:

**Theorem 0.1** (L.). *If there is an inaccessible $\kappa$ which is a limit of $<\kappa$-supercompact cardinals then there is a stationary set preserving forcing $\mathbb{P}$ with*

$$V^{\mathbb{P}} \models \text{“}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

**Convention 0.2.** All ideals in this talk are uniform, normal and on $\omega_1$.

- If $\mathcal{I}$ is an ideal then the associated forcing is $\mathbb{P}_{\mathcal{I}} = \mathcal{P}(\omega_1)/\sim_{\mathcal{I}}$ with the order induced by inclusion. Here, $A \sim_{\mathcal{I}} B$ iff $A \triangle B \in \mathcal{I}$.
- If $G$ is $\mathbb{P}_{\mathcal{I}}$-generic over $V$ then $U_G = \{A \mid [A]_{\sim_{\mathcal{I}}} \in G\}$ is a $V$-ultrafilter which induces the generic ultrapower

$$j_G : V \to \mathrm{Ult}(V, U_G).$$

## 0.1 Precipitous Ideals

**Definition 0.3.** An ideal $I$ is precipitous if: For all generic $G \subseteq \mathbb{P}_{\mathcal{I}}$, $\mathrm{Ult}(V, U_G)$ is wellfounded.

**Theorem 0.4** (Mitchell, [JMMP80]). *A precipitous ideal on $\omega_1$ can be forced from a measurable cardinal.*

Idea: Collapse a measurable to $\omega_1$, the ideal dual to the measure on $\kappa$ then generates a precipitous ideal in the extension.

**Theorem 0.5** (Magidor, [JMMP80]). *“$\mathrm{NS}_{\omega_1}$ is precipitous” can be forced from a measurable cardinal.*

Idea: First collapse measurable to $\omega_1$ as above, then turn the precipitous ideal into $\mathrm{NS}_{\omega_1}$ by killing stationary sets.

This is optimal:

**Theorem 0.6** ([JMMP80]). *The following theories are equiconsistent:*

1. ZFC + “*There is a precipitous ideal*”
2. ZFC + “*There is a measurable cardinal*”

## 0.2 Saturated Ideals

**Definition 0.7.** An ideal $I$ on $\omega_1$ is saturated if $\mathbb{P}_{\mathcal{I}}$ is $\omega_2$-cc.

Saturated ideals are precipitous.

**Theorem 0.8** (Kunen,[Kun78]). *A saturated ideal on $\omega_1$ can be forced from a huge cardinal.*

Idea: Let $j: V \to M$ witness that $\kappa$ is huge. Turn $\kappa$ into $\omega_1$ and $j(\kappa)$ into $\omega_2$ with a special kind of collapse to make lifting arguments work.

**Theorem 0.9** (Steel-Van Wesep,[SVW82]). *"$\mathrm{NS}_{\omega_1}$ is saturated" can be forced over (canonical) models of* $\mathrm{AD} + \mathrm{AC}_{\mathbb{R}}$.

Precursor to $\mathbb{P}_{\max}$.

**Theorem 0.10** (Foreman-Magidor-Shelah,[FMS88]). *"$\mathrm{NS}_{\omega_1}$ is saturated" can be forced from a supercompact cardinal by semiproper forcing.*

This was done by forcing the forcing axiom MM. To each maximal antichain $\mathcal{A}$ of $\mathrm{NS}^+_{\omega_1}$, one associates the sealing forcing $\mathbb{S}_{\mathcal{A}}$ which preserves stationary sets and turns $\mathcal{A}$ into a maximal antichain of size $\leqslant \omega_1$. At this point $\mathcal{A}$ is "sealed", i.e. the maximality of $\mathcal{A}$ cannot be destroyed by further forcing without collapsing $\omega_1$. Applying MM to $\mathbb{S}_{\mathcal{A}}$ shows that $\mathcal{A}$ is sealed to begin with.

Key tool: Iteration of semiproper forcing.

**Definition 0.11.** A forcing $\mathbb{P}$ is proper iff for any large enough regular $\theta$, for any countable $X \prec H_\theta$ with $\mathbb{P} \in X$ and any $p \in \mathbb{P} \cap X$, there is a $(X, \mathbb{P})$-generic condition $q \leqslant p$, that is

$$q \Vdash \check{X}[\dot{G}] \cap V = \check{X}.$$

Countably closed forcings and ccc forcings are proper. Proper forcings are not useful to force "$\mathrm{NS}_{\omega_1}$ is saturated" as proper forcings cannot increase $\delta^1_2$.

**Theorem 0.12** (Woodin,[Woo10]). *Suppose $\mathrm{NS}_{\omega_1}$ is saturated and $\mathcal{P}(\omega_1)^\sharp$ exists. Then $\delta^1_2 = \omega_2$.*

**Definition 0.13.** A forcing $\mathbb{P}$ is semiproper iff for any large enough regular $\theta$, for any countable $X \prec H_\theta$ with $\mathbb{P} \in X$ and any $p \in \mathbb{P} \cap X$, there is a $(X, \mathbb{P})$-semigeneric condition $q \leqslant p$, that is

$$q \Vdash \check{X} \sqsubseteq \check{X}[\dot{G}].$$

Here, $X \sqsubseteq Y$ means $X \subseteq Y$ and $X \cap \omega_1 = Y \cap \omega_1$. Semiproper forcings preserve stationary subsets of $\omega_1$, but can consistently give regular cardinals countable cofinality.

**Theorem 0.14** (Shelah,[She98]). *An RCS-iteration of semiproper forcings is semiproper.*

An RCS iteration is just a countable support iteration that takes into account that there may be new cardinals with countable cofinality in intermediate extensions.

**Theorem 0.15** (Shelah, see [Sch11] for a proof). *"$\mathrm{NS}_{\omega_1}$ is saturated" can be forced from a Woodin cardinal by semiproper forcing.*

Idea: Force with sealing forcings for maximal antichains in $\mathrm{NS}_{\omega_1}^+$, but only if they happen to be semiproper. Otherwise force with $\mathrm{Col}(\omega_1, 2^{\omega_1})$. Use the Woodin cardinal to prove that sealing forcings are semiproper often enough along the way.

This is optimal:

**Theorem 0.16** (Jensen-Steel, [JS13] see also [Ste17])**.** *If there is a saturated ideal then there is an inner model with a Woodin cardinal.*

This follows from the existence of the core model below a Woodin cardinal ("without a measurable") and basic properties of the core model.

## 0.3 Dense Ideals

**Definition 0.17.** An ideal $\mathcal{I}$ is $\omega_1$-dense if $\mathbb{P}_{\mathcal{I}}$ has a dense subset of size $\omega_1$.

- This is equivalent to "$\mathbb{P}_{\mathcal{I}}$ is forcing equivalent to $\mathrm{Col}(\omega, \omega_1)$".
- It follows that if $\mathcal{I}, \mathcal{J}$ are $\omega_1$-dense then $\mathbb{P}_{\mathcal{I}} \cong \mathbb{P}_{\mathcal{J}}$.
- $\omega_1$-dense ideals are saturated.

**Theorem 0.18** (Woodin, see [AST+22] for a proof)**.** *An $\omega_1$-dense ideal can be forced over a canonical model of* $\mathrm{AD}_{\mathbb{R}}$ + "$\Theta$ *is regular*".

This gives a model of ZFC + CH, so the ideal is not $\mathrm{NS}_{\omega_1}$.

**Theorem 0.19** (Shelah, Woodin independently)**.** *If* $\mathrm{NS}_{\omega_1}$ *is $\omega_1$-dense then* CH *fails. In fact,* $2^\omega = 2^{\omega_1}$.

**Theorem 0.20** (Adolf-Sargsyan-Trang-Wilson-Zeman, Woodin, [AST+22])**.** *The following theories are equiconsistent:*

(*i*) ZF + $\mathrm{AD}_{\mathbb{R}}$ + "$\Theta$ *is regular*".

(*ii*) ZFC + CH + *"There is an $\omega_1$-dense ideal".*

**Theorem 0.21** (Woodin, see [For10] for a proof)**.** *If there is an almost huge cardinal then there is an $\omega_1$-dense ideal in a forcing extension.*

This is an improvement of Kunen's argument.

**Theorem 0.22** (Woodin,[Woo10])**.** ZFC + "$\mathrm{NS}_{\omega_1}$ *is $\omega_1$-dense" can be forced over canonical models of* $\mathrm{AD}^+$.

This was achieved with a sibling of $\mathbb{P}_{\max}$ called $\mathbb{Q}_{\max}$.

This is once again optimal:

**Theorem 0.23** (Woodin)**.** *The following theories are equiconsistent:*

(*i*) ZF + AD.

(*ii*) ZFC + *"There is an $\omega_1$-dense ideal".*

(*iii*) ZFC + "$\mathrm{NS}_{\omega_1}$ *is $\omega_1$-dense".*

This theorem was the initial motivation for what is now known as core model induction.

## 0.4 $\mathbb{P}_{\max}$ and $\mathbb{Q}_{\max}$

Suppose $(M,\mathcal{I})$ is a countable structure, $(M;\in,\mathcal{I}) \models \mathrm{ZFC}^- +$ "$\omega_1$ exists" + "$\mathcal{I}$ is a precipitous ideal". (We do not require $\mathcal{I} \in M$, merely amenability.
If $g_0$ is generic over $(M_0,\mathcal{I}_0) = (M,\mathcal{I})$ for $(\mathbb{P}_{\mathcal{I}_0})^(M,\mathcal{I})$ in the sense that $g_0$ hits all maximal antichains definable over $(M;\in,\mathcal{I})$ then get

$$j_0 \colon (M_0;\in,\mathcal{I}_0) \to (M_1;\in,\mathcal{I}_1).$$

Where $(M_1,\mathcal{I}_1) = \mathrm{Ult}((M_0,\mathcal{I}_0),U_{g_0})$. Can iterate this procedure, take direct limit at limit steps.

**Definition 0.24.** $(M,\mathcal{I})$ is generically iterable if all (countable) generic iterates of $(M,\mathcal{I})$ are wellfounded.

**Definition 0.25.** A $\mathbb{P}_{\max}$-condition is of the form $p = (M,\mathcal{I},a)$ where

(*i*) $(M,\mathcal{I})$ is generically iterable.

(*ii*) $M \models \mathrm{MA}_{\omega_1}$

(*iii*) $a \in M$ and $M \models$ "$a \subseteq \omega_1 \wedge \omega_1^{L[a]} = \omega_1$".

$\mathbb{P}_{\max}$ is ordered by $(N,\mathcal{J},b) \leqslant (M,\mathcal{I},a)$ iff there is a generic iteration of $(M,\mathcal{I})$ in $N$ with final map

$$j \colon (M,\mathcal{I},a) \to (M^*,\mathcal{I}^*,a^*)$$

so that $\mathcal{I}^* = \mathcal{J} \cap M^*$ and $a^* = b$.

Point (*ii*) and (*iii*) guarantee that any generic iteration of $(M,\mathcal{I})$ is completely determined by the image of $a$ in the final model. Hence, if $G$ is a $\mathbb{P}_{\max}$-filter then

$$\mathcal{D}_G\{p,\pi_{p,q} \mid q \leqslant p, p,q \in G\}$$

is a directed system where $\pi_{p,q}$ is the unique final iteration map witnessing $q \leqslant p$.
If $\mathrm{AD}^{L(\mathbb{R})}$ holds then $\mathbb{P}_{\max}$ is "self-replicating": if $G$ is $\mathbb{P}_{\max}$-generic then the direct limit $(M_G,\mathcal{I}_G,a_G)$ along $\mathcal{D}_G$ is a "big $\mathbb{P}_{\max}$-condition", i.e. it is a $\mathbb{P}_{\max}$-condition in $V^{\mathrm{Col}(\omega,\omega_1)}$ in which it becomes countable. $M_G$ collects many $\Sigma_1$-truths along the directed system, indeed Woodin shows that the $\Sigma_1$-theory of $M_G$ is maximal (as large as it could be reasonably).

It turns out that

$$(M_G,\mathcal{I}_G) = (H_{\omega_2},\mathrm{NS}_{\omega_1})^{L(\mathbb{R})[G]}.$$

This motivates the following axiom:

**Definition 0.26.** $(*)$ holds if $L(\mathbb{R}) \models \mathrm{AD}$ and there is a $\mathbb{P}_{\max}$-filter $G$ generic over $L(\mathbb{R})$ with

$$(M_G,\mathcal{I}_G) = (H_{\omega_2},\mathrm{NS}_{\omega_1}).$$

**Theorem 0.27** (Woodin,[Woo10])**.** *If $L(\mathbb{R}) \models \mathrm{AD}$ and $G$ is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ then $L(\mathbb{R})[G] \models \mathrm{ZFC}$ + "$\mathrm{NS}_{\omega_1}$ is saturated".*

However, $\mathrm{NS}_{\omega_1}$ is not $\omega_1$-dense in $L(\mathbb{R})^{\mathbb{P}}_{\max}$.

**Definition 0.28.** A $\mathbb{Q}_{\max}$-condition is of the form $p = (M, \mathcal{I}, f)$ where

$(i)$ $(M, \mathcal{I})$ is generically iterable.

$(ii)$ $(M; \in, \mathcal{I}) \models$ "$\mathcal{I}$ is $\omega_1$-dense".

$(iii)$ $f \in M$ and $M \models$ "$f$ witnesses $\diamondsuit^+_{\mathcal{I}}(\omega_1^{<\omega})$" (see Part II, here this means that $f$ codes a dense embedding $\mathrm{Col}(\omega, \omega_1) \to \mathbb{P}_{\mathcal{I}}$).

$\mathbb{Q}_{\max}$ is ordered by $(N, \mathcal{J}, g) \leqslant (M, \mathcal{I}, f)$ iff there is a generic iteration of $(M, \mathcal{I})$ in $N$ with final map

$$j\colon (M, \mathcal{I}, f) \to (M^*, \mathcal{I}^*, f^*)$$

so that $\mathcal{I}^* = \mathcal{J} \cap M^*$ and $f^* = g$.

$\mathbb{Q}_{\max}$ is self-replicating, similar to $\mathbb{P}_{\max}$. Once again, a generic iteration of a $\mathbb{Q}_{\max}$-condition $(M, \mathcal{I}, f)$ is uniquely determined by the final image of $f$.

**Theorem 0.29** (Woodin,[Woo10])**.** *If $L(\mathbb{R}) \models \mathrm{AD}$ and $G$ is $\mathbb{Q}_{\max}$-generic over $L(\mathbb{R})$ then $L(\mathbb{R})[G] \models \mathrm{ZFC}$ + "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense".*

# 1 Part II

## 1.1 The Ansatz

- $\mathbb{Q}_{\max}$-$(*)$ is $(*)$ with $\mathbb{P}_{\max}$ replaced by $\mathbb{Q}_{\max}$. By Asperó-Schindler, $\mathrm{MM}^{++} \Rightarrow (*)$. There should be some forcing axiom FA which solves

$$\frac{\mathrm{MM}^{++}}{(*)} = \frac{\mathrm{FA}}{\mathbb{Q}_{\max}\text{-}(*)}.$$

- So FA implies $\mathbb{Q}_{\max}$-$(*)$ which in turn implies"$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense".

Only known way to force such a strong forcing axiom:

- Iterate small nice-ish forcings up to a supercompact $\kappa$ via a RCS-iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$.
- Invoke an iteration theorem to argue that $\omega_1$ (and suitable additional structure) is preserved along the iteration.
- Employ Baumgartner's argument to get the forcing axiom.

Here, have "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense" in $V^{\mathbb{P}}$ as witnessed by a sequence $\vec{S} = \langle S_i \mid i < \omega_1 \rangle$ of stationary sets. $\mathbb{P}$ is $\kappa$-cc so that already $\vec{S} \in V^{\mathbb{P}_\alpha}$ for some $\alpha < \kappa$.

- Most likely, $\mathrm{NS}_{\omega_1}$ is not $\omega_1$-dense in $V^{\mathbb{P}_\alpha}$.
- But then $\mathbb{P}_{\alpha,\kappa}$ **must kill stationary sets** of $V^{\mathbb{P}_\alpha}$.

  *Proof.* In $V^{\mathbb{P}_\alpha}$ let $T \subseteq \omega_1$ be stationary so that no $S_i$ is below $T$ (i.e. $S_i \backslash T \in \mathrm{NS}^+_{\omega_1}$ for all $i$). There are only two ways to fix this: Either kill $T$, or kill $S_i \backslash T$ for some $i < \omega_1$. Either way involves killing a stationary set. □

- Also $\mathbb{P}_{\alpha,\kappa}$ **must preserve the $\Pi_1$-properties of $\vec{S}$ that hold in $V^{\mathbb{P}}$.**

## 1.2 Iterating while killing stationary sets

**The First Obstacle**

For a stationary $S \subseteq \omega_1$, let $\mathrm{CS}(S)$ denote the forcing that shoots a club through $S$.

- Let $\omega_1 = \bigcup_n S_n$ be a partition into stationary sets.
- Consider the iteration $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ where

$$\Vdash_{\mathbb{P}_n} \dot{\mathbb{Q}}_n = \mathrm{CS}(\omega_1 - \check{S}_n)$$

  (choose your favorite support).

- In $V^{\mathbb{P}}$, $\omega_1^V$ is the countable union of non-stationary sets.
- So $\omega_1^V$ is collapsed.
- Problem: At each step, we go back to $V$ to kill a set from there.
- **Solution: Only kill stationary sets that were just added in the last step!**

**The Second Obstacle**

This is Shelah's example of an iteration of SSP forcings collapsing $\omega_1$ (see [She98]).

- First force a function $g_0 \colon \omega_1 \to \omega_1$ above all canonical functions. Then force some $g_1$ above all canonical functions, but below $g_0$. Continue like this, get

$$\text{canonical functions} < g_n < g_{n-1} < \cdots < g_1 < g_0 \mod \mathrm{NS}_{\omega_1}$$

  at stage $n$. These forcings preserve stationary sets, but not all are semiproper. In the limit $\omega_1$ is collapsed (as there is no infinite decreasing sequence of such functions).
  **Solution: Mostly use forcings with good "regularity properties".**

These are the only two obstacles!

**Theorem 1.1** (L.). *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a RCS-iteration of $\omega_1$-preserving forcings and assume that for all $\alpha < \gamma$:*

- $\Vdash_{\mathbb{P}_{\alpha+1}}$ SRP
- $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ *preserves stationary sets from* $\bigcup_{\beta<\alpha} V[\dot{G}_\beta]$"

*Then $\mathbb{P}$ preserves $\omega_1$.*

This is a "cheapo iteration theorem", but good enough for our purposes.

SRP hides the relevant regularity property. What is it?

For now consider an iteration $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ iteration of length $\omega$ of $\omega_1$-preserving forcings that do not kill "old stationary sets".

- Want to argue somehow that $\mathbb{P}$ preserves $\omega_1$.
- So must find countable $X \prec H_\theta$ and $p$ so that

$$p \Vdash \check{X} \sqsubseteq \check{X}[\dot{G}].$$

Let $X \prec H_\theta$ countable with $\mathbb{P} \in X$. Want to find $p_n \in \mathbb{P}_n$ so that $(p_n)_{n<\omega}$ is decreasing in $\mathbb{P}$ and

$$p_n \Vdash_{\mathbb{P}_n} \check{X} \sqsubseteq \check{X}[\dot{G}_n].$$

Suppose in step $n$ of this argument, have

- Next forcing $\mathbb{Q} = \dot{\mathbb{Q}}_n^{G_n}$
- $S \subseteq \omega_1$ is stationary, $S \in X[G_n]$ but $\Vdash_{\mathbb{Q}} \check{S} \in \mathrm{NS}_{\omega_1}$ and
- $\delta^{X[G_n]} := X[G_n] \cap \omega_1 \in S$.

Then there is no way to continue! Must avoid this at all cost!

So need to start with $X$ which avoids this problem, i.e. if $S \in X$ and $\mathbb{Q}_0$ kills $S$ then $\delta^X \notin S$. This is easily possible!

Our regularity property should hand us some $p_0 \in \mathbb{Q}_0$ with

$$p_0 \Vdash_{\mathbb{Q}_0} \check{X} \sqsubseteq \check{X}[\dot{G}_1].$$

Even then, we might end up with the same problem at the next step $X[G_1]$! So $p_0$ must moreover avoid this situation for $X[G_1]$!

**Definition 1.2.** Say that a countable $Y \prec H_\theta$ respects an ideal $\mathcal{I}$ if $\delta^Y \notin S$ whenever $S \in \mathcal{I} \cap Y$.

In other words, need that $X[G_1]$ respects the ideal $\{S \subseteq \omega_1 \mid \mathbb{Q}_1 \text{ kills } S\}$.

**Definition 1.3.** Suppose $\mathbb{Q}$ is $\omega_1$-preserving forcing. $\mathbb{Q}$ is **respectful** if: Whenever

- $Y \prec H_\theta$ countable, $\mathbb{Q} \in Y$, $p \in \mathbb{Q} \cap Y$
- $\dot{I} \in Y$ is a $\mathbb{Q}$-name for an ideal on $\omega_1$.

Then one of the following:

1. There is $q \leqslant p$ and $q$ forces

$$Y \sqsubseteq Y[G] \wedge Y[G] \text{ respects } \dot{I}^G$$

2. Or: $Y$ does **not** respect $\dot{I}^p := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \dot{I}\}$.

This is a very strong regularity property! If $\mathbb{Q}$ is respectful and preserves stationary sets then $\mathbb{Q}$ is semiproper, but semiproper forcings need not be respectful.

Let's get back to our toy problem. Start with $X \prec H_\theta$ with $\mathbb{P} \in X$ so that $X$ respects $\{S \subseteq \omega_1 \mid \mathbb{Q}_0 \text{ kills } S\}$.
Let $\dot{I}$ be the $\mathbb{Q}_0$-name for

$$\{S \subseteq \omega_1 \mid \dot{\mathbb{Q}}_1^{G_1} \text{ kills } S\}.$$

Since $\dot{\mathbb{Q}}_1^{G_1}$ does not kill old sets, $X$ **trivially respects** $\dot{I}^{\mathbb{1}_{\mathbb{Q}_0}} \subseteq V$.
If $\mathbb{Q}_0$ is respectful then find $p_0$ so that

$$p_0 \Vdash_{\mathbb{Q}_0} \check{X} \sqsubseteq \check{X}[\dot{G}_1] \wedge \check{X}[\dot{G}_1] \text{ respects } \dot{I}^{\dot{G}_1}.$$

We are back in the same situation, only one step further. Can chain these arguments together!

**Lemma 1.4.** *If $\mathbb{P}$ is a countable support iteration of respectful forcings which do not kill old stationary sets then $\mathbb{P}$ preserves $\omega_1$.*

Unfortunately, RCS iterations of respectful forcings need not be respectful. But we can simply nuke this problem!

**Theorem 1.5** (L.)**.** *If* SRP *holds then every $\omega_1$-preserving forcing is respectful.*

*Proof.* Let $\mathbb{Q}$ be $\omega_1$-preserving, $Y \prec H_\theta$, $q \in \mathbb{Q} \cap Y$, $\dot{I} \in Y$ as in definition. Have to show:

- Either there is $r \leqslant q$ forcing $Y \sqsubseteq Y[G]$ respects $\dot{I}^G$
- or $Y$ does not respect $\dot{I}^q$.

Let $\mu = (2^{|Q|})^+ \in Y$ and $\mathcal{S} = \{Z \prec H_\mu \mid \nexists r \leqslant q \text{ forcing "} Z \sqsubseteq Z[G] \text{ respects } \dot{I}^G\text{"}\} \in Y$.
By SRP, can find continuous increasing $\vec{Z} = \langle Z_\alpha \mid \alpha < \omega_1 \rangle \in Y$ s.t.:

- $\mathbb{Q}, q, \dot{I} \in Z_0$
- $Z_\alpha \prec H_\mu$

- Either $Z_\alpha \in \mathcal{S}$ or there is no $Z_\alpha \sqsubseteq Z$ with $Z \in \mathcal{S}$.

Let $G \subseteq \mathbb{Q}$ generic, $q \in G$. Let $S = \{\alpha < \omega_1 \mid Z_\alpha \in \mathcal{S}\}$.

**Claim 1.6.** $S \in I := \dot{I}^G$.

*Proof.* Suppose otherwise, $S \in I^+$. $\langle Z_\alpha[G] \mid \alpha < \omega_1\rangle$ is continuous increasing sequence of elementary substructures of $H_\mu^{V[G]}$. Find club $C \subseteq \omega_1$ with $\alpha = \delta^{Z_\alpha} = \delta^{Z_\alpha[G]}$. For any $\alpha \in S \cap C$, can find $T_\alpha \in I \cap Z_\alpha[G]$ with $\alpha = \delta^{Z_\alpha[G]} \in T_\alpha$. By normality of $I$, there is $S_0 \subseteq S \cap C$ in $I^+$ and $T$ so that $T_\alpha = T$ for $\alpha \in S_0$. But then $S_0 \subseteq T$, contradicting $T \in I$. □

Case 1: $\delta^Y \in S$. As $S \in \dot{I}^q \cap Y$, $Y$ does not respect $\dot{I}^q$.

Case 2: $\delta^Y \notin S$. As $Z_{\delta^Y} \sqsubseteq Y \cap H_\mu$, $Y \cap H_\mu \notin \mathcal{S}$. Thus there is $r \leqslant q$ forcing $Y \sqsubseteq Y[G]$ and $Y[G]$ respects $\dot{I}^G$. □

**Remark 1.7.** In $L$, $\mathrm{Add}(\omega_1, 1)$ is *not* respectful.

### 1.3 $\diamondsuit(\omega_1^{<\omega})$

Recall that we first force a candidate $\langle S_i \mid i < \omega_1\rangle$ which might witness "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense" in the future. This cannot be any random collection of $\omega_1$-many stationary sets.

**Lemma 1.8** (Tennenbaum (?))**.** *If $\mathbb{P}$ is a forcing of size $\omega_1$ which collapses $\omega_1$ then there is a dense embedding $\pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}$.*

- $\Rightarrow$ Better: First force a candidate $\pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathcal{P}(\omega_1)\backslash\mathrm{NS}_{\omega_1}$. In the end, want $\dot{[]}_{\mathrm{NS}_{\omega_1}} \circ \pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ a dense embedding.
- This suggests we should isolate properties of $\pi$, and then iterate forcing preserving these properties of $\pi$.

**Definition 1.9** (Woodin)**.** $\diamondsuit(\omega_1^{<\omega})$ holds if there is an embedding $\pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathcal{P}(\omega_1)\backslash\mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathrm{Col}(\omega, \omega_1)$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathrm{Col}(\omega, \omega_1) \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

**Lemma 1.10.** *Suppose $[\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ is a dense embedding. Then $\pi$ witnesses $\diamondsuit(\omega_1^{<\omega})$.*

*Proof Sketch.* Let $p \in \mathrm{Col}(\omega, \omega_1)$, $X \prec H_{\omega_2}$ countable so that $\omega_1 \cap X =: \delta^X \in \pi(p)$. Let $A \subseteq \mathrm{Col}(\omega, \omega_1)$, $A \in X$, be a maximal antichain. $\Rightarrow \mathcal{A} := [\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi[A] \subseteq \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ is a max. antichain, thus $\triangle\mathcal{A}$ contains a club $C \in X$, so $\delta^X \in C$. It follows that there is $q \in X \cap A$ with $\delta^X \in \pi(q)$. □

More generally $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$

**Definition 1.11.** Let $\mathbb{B} \subseteq \omega_1$ be a forcing. $\diamondsuit(\mathbb{B})$ holds if there is an embedding $\pi \colon \mathbb{B} \to \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathbb{B}$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathbb{B} \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

We call such $X$ $\pi$-slim.
The stronger $\diamondsuit^+(\mathbb{B})$ holds if there is $\pi$ witnessing $\diamondsuit(\mathbb{B})$ so that every $X \prec H_\theta$ with $f, \mathbb{B} \in X$ is $\pi$-slim.

**Lemma 1.12.** *If $\diamondsuit$ holds then $\diamondsuit(\mathbb{B})$ holds for every forcing $\mathbb{B} \subseteq \omega_1$ (but not necessarily $\diamondsuit^+(\mathbb{B})$).*

# 2 Part III

**Lemma 2.1** (Essentially Woodin,[Woo10])**.** *$\pi \colon \mathbb{B} \to \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ witnesses $\diamondsuit(\mathbb{B})$ iff $[\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi \colon \mathbb{B} \to (\mathbb{P}_{\mathrm{NS}_{\omega_1}})^W$ is a complete embedding in some outer model $W$.*

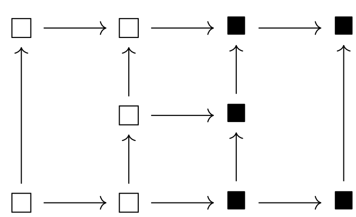
**Definition 2.2.** QM is the axiom: $\exists \pi$ witnessing $\diamondsuit(\omega_1^{<\omega})$ so that

$$\mathrm{FA}_{\omega_1}(\{\mathbb{P} \mid V^{\mathbb{P}} \models \text{“}\pi \text{ witnesses } \diamondsuit(\omega_1^{<\omega})\text{”}\})$$

holds.

QM implies...

- there is a Suslin tree,
- “almost disjoint coding” fails,
- the Cichon diagram is



- SRP $\wedge$ $\neg$MRP.

As a consequence, we also get the following as soon as we show QM to be consistent:

**Corollary 2.3.** SRP *does not imply* MRP*.*

This may be somewhat surprising as roughly speaking $\frac{\mathrm{SRP}}{\mathrm{MM}} = \frac{\mathrm{MRP}}{\mathrm{PFA}}$ and clearly MM $\Rightarrow$ PFA.

**Lemma 2.4.** QM *implies* $\mathrm{NS}_{\omega_1}$ *is* $\omega_1$*-dense!*

*Proof Sketch.* Let $\pi$ witness $\diamondsuit(\omega_1^{<\omega})$. Want to show that $\pi$ is a dense embedding. If not, let $S \in \mathrm{NS}_{\omega_1}^+$ with no set in $\mathrm{ran}(\pi)$ below $S$. Can show that $\mathrm{CS}(\omega_1 - S)$ is $\pi$-preserving.

**Claim 2.5.** $\mathrm{CS}(\omega_1 - S)$ *is $\pi$-preserving.*

*Proof.* Let $r \in \mathrm{CS}(\omega_1 - S)$, $p \in \mathrm{Col}(\omega, \omega_1)$ and $\dot{C}$ a name for a club in $[H_{\omega_2}^V[G]]^\omega$. We have to show that if $G$ is generic with $r \in G$ then there is a $\pi$-slim $X \prec H_{\omega_2}^{V[G]}$ in $\mathcal{C}$ with $X \cap \omega_1 \in \pi(p)$.

As $\pi(p) \nsubseteq T \mod \mathrm{NS}_{\omega_1}$, we can find some countable $X \prec H_\theta$ with $X \cap \omega_1 \in \pi(p) \setminus T$ so that $X$ contains all relevant parameters. Let $M_X$ be the transitive collapse of $X$. As $X$ is $\pi$-slim,

$$g = \{q \in \mathrm{Col}(\omega, \omega_1)^{M_X} \mid \omega_1 \cap X \in \pi(q)\}$$

is generic over $M_X$. We can now build a generic sequence over $M_X[g]$ starting with $r$. As $\omega_1^{M_X} \notin T$, this sequence has a lower bound $r_*$ and $r_*$ forces $X[G]$ to be $\pi$-slim (essentially by the product lemma). Clearly $r_*$ forces $X[G]$ to be in $\mathcal{C}$ as well. ∎

But by QM applied to $\mathrm{CS}(\omega_1 - S)$, $H_{\omega_2} \prec_{\Sigma_1} (H_{\omega_2})^{V^{\mathrm{CS}(\omega_1 - S)}}$, contradiction. ∎

The real challenge is to force QM.

**Definition 2.6.** Suppose $\pi$ witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is **$\pi$-proper** if: Whenever

- $X \prec H_\theta$ countable and $\pi$-slim, $\mathbb{P} \in X$
- $p \in \mathbb{P} \cap X$

Then there is $(X, \mathbb{P}, \pi)$-generic $q \leqslant p$, i.e. forces

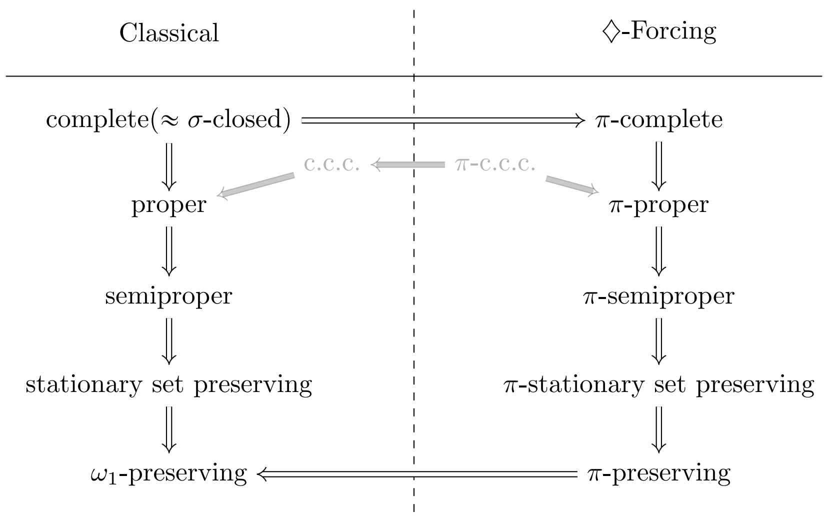
$$X = X[G] \cap V \wedge X[G] \text{ is } \pi\text{-slim.}$$

Analogously, define $\pi$-semiproperness.

**Definition 2.7.** Suppose $\pi$ witnesses $\diamondsuit(\mathbb{B})$. A set $S \subseteq \omega_1$ is $\pi$-stationary if for large enough regular $\theta$ and all clubs $\mathcal{C} \subseteq [H_\theta]^\omega$ there is some $\pi$-slim $X \in \mathcal{C}$, $X \prec H_\theta$ with $\delta^X \in S$.

| $\mathbb{B} = ...$ | $\{\mathbb{1}\}$ | $T$ a Suslin tree |
|---|---|---|
| $\pi$-proper is... | proper | proper + $T$-preserving |
| $\pi$-semiproper is... | semiproper | semiproper + $T$-preserving |

| $\mathbb{B} = ...$ | Cohen forcing |
|---|---|
| $\pi$-proper is... | "proper for a weakly Luzin sequence" |
| $\pi$-semiproper is... | "semiproper for a weakly Luzin sequence" |




We really only care about $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$.
Suppose $\pi$ witnesses $\diamondsuit(\mathbb{B})$.

**Theorem** *Countable support iterations of $\pi$-proper forcings are $\pi$-proper*

**Theorem** *RCS iterations of $\pi$-semiproper forcings are $\pi$-semiproper.*

**Corollary 2.8** (Shelah,[She98])**.** *Proper (semiproper) forcings are closed under countable (RCS) support iterations.*

**Corollary 2.9** (Essentially Miyamoto[Miy93],[Miy02])**.** *Suppose $T$ is a Suslin tree. Proper (semiproper) + $T$-preserving forcings are closed under countable (RCS) support iterations.*

We only want to iterate $\pi$-semiproper forcings here for $\pi$ a witness of $\diamondsuit(\omega_1^{<\omega})$.

**Corollary 2.10.** *If there is a supercompact cardinal then there is a $\pi$-semiproper (and hence $\pi$-preserving) poset forcing* SRP*.*

**Corollary 2.11.** *If there is a Woodin cardinal then there is a $\pi$-semiproper (and hence $\pi$-preserving) poset forcing "$\mathrm{NS}_{\omega_1}$ is saturated".*

Forcing QM
To force QM we need to

- force a witness $\pi$ of $\diamondsuit(\omega_1^{<\omega})$ (easy)
- and then iterate arbitrary $\pi$-preserving forcings and preserve $\pi$ (hard).
- Iterating $\pi$-semiproper forcings gives the forcing axiom for all $\pi$-stationary set preserving forcings, but that is not enough!

The iteration theorem from Part II generalizes.

**Theorem 2.12.** *Suppose $\mu$ witnesses $\diamondsuit(\mathbb{B})$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a RCS-iteration of $\mu$-preserving forcings and assume that for all $\alpha < \gamma$:*

- $\Vdash_{\mathbb{P}_{\alpha+1}}$ SRP
- $\Vdash_{\mathbb{P}_\alpha}$ "*$\dot{\mathbb{Q}}_\alpha$ preserves $\mu$-stationary sets from $\bigcup_{\beta<\alpha} V[\dot{G}_\beta]$*"

*Then $\mathbb{P}$ preserves $\mu$.*

We need to get around the restriction of preserving old stationary sets. Suppose $\pi$ witnesses $\diamondsuit(\omega_1^{<\omega})$.

**Definition 2.13.** A $Q$-iteration is a RCS iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ of $\pi$-preserving forcings so that for all $\alpha < \gamma$

- $\Vdash_{\mathbb{P}_\alpha+2}$ SRP
- $\Vdash_{\mathbb{P}_\alpha+1}$ "$\dot{\mathbb{Q}}_{\alpha+1}$ makes $\pi$ dense for sets in $V[\dot{G}_{\alpha+1}]$".

**Corollary 2.14** (Work-Life-Balance Theorem)**.** *$Q$-iteration preserve $\pi$.*

This means we can force QM from large cardinals provided we find the $\dot{\mathbb{Q}}_{\alpha+1}$ which make "$\pi$ dense for ground model sets" ("sealing forcings for $\omega_1$-density").

## 2.1 The New Sealing Forcing

MM$^{++}$ $\Rightarrow$ $(*)$ Assuming $H_{\omega_2}$ is a "big $\mathbb{P}_{\max}$-condition", Asperó-Schindler construct a forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$ the following picture exists:

$$
\begin{array}{ccccc}
 & & D^* & & \\
 & & \Cup & & \\
 & & q_0 & \xrightarrow{\quad \sigma_{0,\omega_1} \quad} & q_{\omega_1} = (N^*, I^*, b^*) \\
 & & \Cup & & \Cup \\
p_0 & \xrightarrow{\quad \mu_{0,\omega_1^N} \quad} & p_{\omega_1^N} & \xrightarrow{\quad \mu_{\omega_1^N,\omega_1} \quad} & p_{\omega_1} \\
\Cap & & & & \parallel \\
\mathbb{P}_{\max} & & & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, A)
\end{array}
$$

- $\mu_{0,\omega_1^N}$ witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$ and $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is *correct* in $V^{\mathbb{P}}$, i.e. $I^* = (\mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N^*$.

Modifications We want to replace $\mathbb{P}_{\max}$ by $\mathbb{Q}_{\max}$. Immediate problem: Then we have to assume that $(H_{\omega_2}, \mathrm{NS}_{\omega_1})$ is (part of) a big $\mathbb{Q}_{\max}$-condition. So $\mathrm{NS}_{\omega_1}$ must already be $\omega_1$-dense!

**Definition 2.15.** $\mathbb{Q}^-_{\max}$-conditions are of the form $(M, I, \pi)$ with:

- $(M, I)$ is generically iterable.
- $M \models$"$\pi$ witnesses $\diamondsuit^+_I(\omega_1^{<\omega})$"

$q = (N, J, \tau) <_{\mathbb{Q}^-_{\max}} (M, I, \pi) = p$ iff in $N$ there is a generic iteration (map) $j : p \to p^* = (M^*, I^*, \pi^*)$ such that:

- $\pi^* = \tau$
- $\tau$ is dense for sets in $M^*$, i.e. if $S \in \mathcal{P}(\omega_1)^{M^*}$ then
  - either $S \in J$
  - or $\exists p \in \mathrm{Col}(\omega, \omega_1^N)\ \tau(p) \subseteq S \mod J$.

$\mathbb{Q}_{\max}$ embeds densly into $\mathbb{Q}^-_{\max}$ (assuming $\mathrm{AD}^{L(\mathbb{R})}$).

Does it work now? We can force $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \pi)$ to be a "big $\mathbb{Q}^-_{\max}$-condition" using $\pi$-semiproper forcing. Following Asperó-Schindler, we get:

$$\begin{array}{ccccc}
 & & q_0 & \xrightarrow{\quad\sigma_{0,\omega_1}\quad} & q_{\omega_1} = (N^*, I^*, \tau^*) \\
 & & \Cup & & \Cup \\
p_0 & \xrightarrow{\quad\mu_{0,\omega_1^N}\quad} & p_{\omega_1^N} & \xrightarrow{\quad\mu_{\omega_1^N,\omega_1}\quad} & p_{\omega_1} \\
\Cap & & & & \| \\
\mathbb{Q}^-_{\max} & & & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, \pi)
\end{array}$$

- $\mu_{0,\omega_1^N}$ witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$ and $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is *correct* in $V^{\mathbb{P}}$, i.e. $I^* = (\mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N^*$.

So $\mathbb{P}$ makes $\pi$ dense for sets in $V$, great! But this it preserve $\pi$? Unclear!!

$\diamondsuit$-Iterations

**Definition 2.16.** A generic iteration $\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$ is a $\diamondsuit$-iteration if: For any sequence $\langle D_i \mid i < \omega_1 \rangle$ of dense subsets of $(\mathcal{P}(\omega_1)^{M_{\omega_1}}/I_{\omega_1})^+$ and any $S \in I^+_{\omega_1} \cap M_{\omega_1}$ have

$$\{\alpha \in S \mid \forall i < \alpha\ U_\alpha \cap \mu^{-1}_{\alpha,\omega_1}[D_i] \neq \varnothing\} \in \mathrm{NS}^+_{\omega_1}$$

where $U_\alpha$ is the generic ultrafilter applied to $M_\alpha$.

All $\diamondsuit$-iterations are correct in the sense that if $(M^*, \mathcal{I}^*)$ is the final model of a $\diamondsuit$-iteration then $\mathcal{I}^* = \mathrm{NS}_{\omega_1} \cap M^*$. But more structure is preserved now! E.g. if $T \in M^*$ is a Suslin tree in $M^*$ then $T$ is really Suslin.

Even better:

**Lemma 2.17.** *Suppose* $(M^*, \mathcal{I}^*)$ *is the final model of a* $\diamondsuit$*-iteration. If*

$$(M^*; \in, \mathcal{I}^*) \models \text{“}\pi \textit{ witnesses } \diamondsuit^+_{\mathcal{I}^*}(\mathbb{B})$$

*then* $\pi$ *witnesses* $\diamondsuit(\mathbb{B})$ *in* $V$*.*

**Theorem 2.18** (L.)**.** *Can modify Asperó-Schindler's* $\mathbb{P}$ *to* $\mathbb{P}_\diamondsuit$ *so that in* $V^{\mathbb{P}_\diamondsuit}$ *the same picture as before exists and* $q_0 \to q_{\omega_1}$ *is a* $\diamondsuit$*-iteration in* $V^{\mathbb{P}_\diamondsuit}$*.*

This is the final piece! We can get our sealing forcings from Woodin cardinals!

**Corollary 2.19.** QM *implies* $\mathbb{Q}_{\max}$-$(*)$*.*

**Theorem 2.20.** *If there is a supercompact limit of supercompact cardinals then* QM *holds in a stationary set preserving forcing extension.*

*Proof Sketch.* First force with $\mathrm{Col}(\omega_1, 2^{\omega_1})$. In the extension, we have a witness $\pi$ of $\diamondsuit(\omega_1^{<\omega})$.

Do a $Q$-iteration up to a supercompact cardinal. If this cardinal is a limit of supercompacts as well, we have enough fuel to constantly force SRP via $\pi$-semiproper forcing. To make the new sealing forcing work, we only need Woodin cardinals. If we picked $\pi$ carefully, the whole iteration will preserve stationary sets from $V$ (collapsing $2^{\omega_1}$ makes this possible). □

**Theorem 2.21.** *If there is an inaccessible* $\kappa$ *which is a limit of* $<\kappa$*-supercompact cardinals then there is a stationary set preserving* $\mathbb{P}$ *with*

$$V^{\mathbb{P}} \models \text{“}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

*Proof.* Pick $\pi$ as before in $V[g]$, an extension by $\mathrm{Col}(\omega, 2^{\omega_1})$. Then force over $V[G]$ with an iteration $\mathbb{P}$ which is a proper class length $Q$-iteration from the perspective of $V[g]_\kappa$. □

## 2.2 The Mystery

How much can the large cardinal assumption of the main theorem be reduced? We used

- an inaccessible on the top to “catch our tail”,
- Woodin cardinals for the “new sealing forcing” and
- (partial) supercompact to satisfy the greedy iteration theorem.

If we could do without SRP, we could plausibly lower the assumption to an inaccessible limit of Woodin cardinals!

**Theorem 2.22** (Woodin,[Woo])**.** *The large cardinal assumption of the main theorem cannot be reduced to an inaccessible limit of Woodin cardinals. In fact, consistently there is a model with an inaccessible limit of Woodin cardinals but no* $\omega_1$*-preserving poset forcing “*$\mathrm{NS}_{\omega_1}$ *is* $\omega_1$*-dense”.*

*Proof.* Work in the least inner model $M$ with an inaccessible limit of Woodin cardinals and a proper class of Woodin cardinals. Suppose $M[G] \models$ "$\mathrm{NS}_{\omega_1}$ is $\omega_1$-dense" and $\omega_1^M = \omega_1^{M[G]}$.

We show that in an extension of $M[G]$, there are divergent models of AD (theorem then follows from gap in consistency strengths). In $M$, we have $\heartsuit$ :

$$\forall \alpha < \omega_1 \exists x \in \mathbb{R}\ (x \text{ codes } \alpha \wedge x \in \mathrm{OD}^{L(A,\mathbb{R})} \text{ for some } A \in \mathrm{uB}) \qquad (\heartsuit)$$

Why? Let $\beta < \omega_1$ so that $M\|\beta \ni x$ some code for $\alpha$. For $\Sigma = (\omega, \omega_1, \omega_1)$-iteration strategy for $M\|\beta$, have $x \in \mathrm{OD}^{L(\Sigma,\mathbb{R})}$.
Note that $\heartsuit$ still holds in $M[G]$! Let $g$ be $M[G]$-generic for $\mathbb{P}_{\mathrm{NS}_{\omega_1}} \cong \mathrm{Col}(\omega, \omega_1)$. We get a generic embedding

$$j_g \colon M[G] \to N.$$

By $\heartsuit$ in $N$, let $x$ code $\omega_1^M$, $x \in \mathrm{OD}^{L(A,\mathbb{R}^N)}$, $L(A, \mathbb{R}^N) \models$ AD. Now, $\mathbb{R}^N = \mathbb{R}^{M[G][g]}$. If there are no divergent models in $M[G][g]$ then $L(A, \mathbb{R}^N)$ is definable in $M[G][g]$ from $\Theta^{L(A,\mathbb{R}^N)}$. But then $x$ is $\mathrm{OD}^{M[G][g]}$, so $x \in M[G]$ by homogeneity of $\mathrm{Col}(\omega, \omega_1)$, contradiction! □

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